UDC 534

PERIODIC SOLUTIONS OF PIECEWISE-CONTINUOUS SYSTEMS WITH A SMALL PARAMETER

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An attempt is made to extend the local Liapunov-Poincare small-parameter theory to piecewise-continuous systems of general form. The problem is reduced to the investigation of an infinite ordered sequence of continuous dynamic systems, which, in a specified sense, is analogous to the original piecewise-continuous system for the study of the motions with a given order of switchings. The solutions are constructed in the form of series in powers of a small parameter and, in contrast to [1], the successive approximations to the switching instants are found after the construction of the corresponding approximations to the true solution. The conditions for the existence and the stability in-the-small of the periodic solutions are obtained in a form very similar to that presented in [2]. In contrast to the method of point mappings [3], the proposed method is not connected with the integration of the exact equations of motion within the intervals of continuity and with the analysis of the successor function [1] obtained after such an integration. It is to be noted that in the particular case when the order of the system does not change during the switchings and the switching equations do not depend on the parameter, the existence conditions derived below turn into those obtained previously in [4].

1. Statement of the problem. On the differentiation of the solution with respect to a parameter. We consider an infinite ordered sequence of dynamic systems

$$x_i^{\cdot} = X_i(x_i, t, \mu), \qquad i = \ldots, -1, 0, 1, \ldots \tag{1.1}$$

where x_i is a $k_i \times 1$ vector. We assume that if x_i belongs to some region G_i of the characteristic phase space, and $0 < \mu < \mu_0$, then the $k_i \times 1$ vector-valued function X_i is analytic in all its arguments and the integral trajectories determined in accordance with (1.1) first intersect the hypersurface

$$g_{i+1}(x_i, t, \mu) = 0 \tag{1.2}$$

at some instant $t = t_{i+1}$. At this instant we assume the presence of a strict correspondence between the dynamic states of the i th and the $(i+1)$st systems, characterized by the equality

$$x_{i+1} = \Phi_{i+1}(x_i, t, \mu) \tag{1.3}$$

where $x_{i+1} \in G_{i+1}$. The scalar g_{i+1} and k_{i+1}-dimensional vector-valued function Φ_{i+1} are also analytic in all their arguments inside G_i. Finally, we assume that a positive integer n exists, guaranteeing the fulfillment of the equalities

$$k_i \equiv k_{i+n}, \qquad X_i(x_i, t, \mu) \equiv X_{i+n}(x_i, t+T, \mu)$$
$$g_{i+1}(x_i, t, \mu) \equiv g_{i+n+1}(x_i, t+T, \mu)$$
$$\Phi_{i+1}(x_i, t, \mu) \equiv \Phi_{i+n+1}(x_i, t+T, \mu) \tag{1.4}$$

where T is some positive constant.

The solving of the successive continuous systems (1.1) under conditions (1.2) and (1.3) permits us to make a judgement on the qualitatively defined motions of the corresponding piecewise-continuous variable-structure system with n essentially distinct switchings. The vector x defining the position of the piecewise-continuous system in such a motion at an arbitrary instant, has a number of components depending on the number i of the interval of continuity and

$$x = x_i, \qquad t_i < t < t_{i+1}, \qquad i = \ldots, -1, 0, 1, \ldots \tag{1.5}$$

The answer to the question on the mutual correspondence of the individual components of vectors x_i and x_{i+1} is always obvious from physical considerations. For what follows it is essential that in contrast to x the vector x_i does not undergo a discontinuity at the instants t_i and t_{i+1} and that it is continuous, in general, for any real t.

Suppose that among the numbers $k_0, k_1, \ldots, k_{n-1}$ the smallest one is k_0 $(k_0 \leqslant k_i)$. Then the posing of the initial conditions

$$g_0\big|_{t=t_*} = a(\mu) \qquad (a \in G) \tag{1.6}$$

where the instant t_* may possibly lie outside the interval (t_0, t_1), allows us to determine uniquely a piecewise-continuous solution x (see (1.5)) for any real t. The specifying at the initial instant t_* of some vector x_j such that $k_j > k_0$ does not allow us, because (1.3) is not invertible, to continue x to the side of decreasing t farther than some $t_l < t_j$ for which first $k_{l-1} < k_l$. The subfamily of solutions of (1.1), continuable to both sides with respect to t, which are determined by initial conditions of type (1.6), evidently, include within itself all the T-periodic solutions which by virtue of (1.5) satisfy the equalities

$$x_i(t, \mu) = x_{i+n}(t+T, \mu) \tag{1.7}$$

We note that as t grows all the other solutions arrive into the subfamily mentioned at the end of a finite interval.

Further, using the techniques of generalized functions [5], we shall write the equations of motion of the successive dynamic systems in the form

$$x_i^{\cdot} = F_i(x_i, x_{i-1}, t, \mu), \qquad F_i = X_i\sigma(g_i) + \Phi_i\sigma^{\cdot}(g_i) \tag{1.8}$$

Here $\sigma(g_i)$ is the unit step defined by the formula

$$\sigma(g_i) = \begin{cases} 0, & g_i < 0 \quad (t < t_i) \\ 1, & g_i > 0 \quad (t > t_i) \end{cases} \tag{1.9}$$

$$\sigma^{\cdot}(g_i) = \delta(g_i)g_i^{\cdot} = \delta(t - t_i) \tag{1.10}$$

(Here, without loss of generality, we assume that the function $g_i(x_{i-1}, t, \mu)$ is negative for $t < t_i$ and positive for $t > t_i$.) The derivative $\sigma^{\cdot}(g_i)$ can also be treated as if it were taken relative to the preceding $(i-1)$st system, so that

$$\sigma^{\cdot}(g_i)=\left(\frac{\partial g_i}{\partial x_{i-1}}X_{i-1}+\frac{\partial g_i}{\partial t}\right)\delta(g_i)$$

We need to construct the solutions of systems (1.8), assuming that $x_i=0$ for $t<t_i$.

Let us now assume that when $\mu=0$ the successive systems (1.8) admit of a family of solutions

$$x_i^{(0)}=\varphi_i(t,\ h_1,\ldots,\ h_s) \tag{1.11}$$

T-periodic in the sense of (1.7) and depending on s arbitrary parameters $h_1,\ldots,h_s$ $(s<k_0)$. The fundamental task of the subsequent investigation is to determine the conditions under which the sequence (1.8) with $\mu\neq 0$ admits of T-periodic solution which turns, as $\mu\to 0$, into one of the solutions in family (1.11), to work out an algorithm for the construction of such a solution for sufficiently small values of the parameter, and also to establish criteria for its stability in-th-small.

Before we proceed to the solution of this problem, let us derive the equation for the derivative

$$u_i(t,\mu)=\partial x_i(t,\mu)/\partial\mu \tag{1.12}$$

To do this we write out the result of a formal differentiation with respect to μ of both sides of Eq. (1.8),

$$u_i^{\cdot}=\frac{\partial' X_i}{\partial\mu}\sigma(g_i)+X_i\delta(g_i)\frac{\partial' g_i}{\partial\mu}+\frac{\partial'\Phi_i}{\partial\mu}\sigma^{\cdot}(g_i)+\Phi_i\frac{d}{dt}\left[\delta(g_i)\frac{\partial' g_i}{\partial\mu}\right] \tag{1.13}$$

Here the prime denotes the "total" partial differentiation, so that, for examole,

$$\frac{\partial' X_i}{\partial\mu}=\frac{\partial X_i}{\partial x_i}u_i+\frac{\partial X_i}{\partial\mu} \tag{1.14}$$

With due regard to (1.10) and also to the fact that the terms

$$\Phi_i\frac{d}{dt}\left[\delta(g_i)\frac{\partial' g_i}{\partial\mu}\right],\qquad -\Phi_i^{\cdot}\delta(g_i)\frac{\partial' g_i}{\partial\mu}$$

ensure one and the same jumps in the components of vector u_i at the instant $t=t_i$, Eq. (1.13) can be rewritten as

$$u_i^{\cdot}=\frac{\partial' X_i}{\partial\mu}\sigma(g_i)+\left[\frac{\partial'\Phi_i}{\partial\mu}-(\Phi_i^{\cdot}-X_i)\frac{\partial' g_i}{\partial\mu}(g_i^{\cdot})^{-1}\right]\sigma^{\cdot}(g_i) \tag{1.15}$$

or, equivalently, as

$$u_i^{\cdot}=A_iu_i\sigma(g_i)+B_iu_{i-1}\sigma^{\cdot}(g_i)+\frac{\partial X_i}{\partial\mu}\sigma(g_i)+Y_i\sigma^{\cdot}(g_i) \tag{1.16}$$

$$A_i=\frac{\partial X_i}{\partial x_i},\qquad B_i=\frac{\partial\Phi_i}{\partial x_{i-1}}-(\Phi_i^{\cdot}-X_i)\frac{\partial g_i}{\partial x_{i-1}}(g_i^{\cdot})^{-1}$$

$$Y_i=\frac{\partial\Phi_i}{\partial\mu}-(\Phi_i^{\cdot}-X_i)\frac{\partial g_i}{\partial\mu}(g_i^{\cdot})^{-1}$$

Here A_i and B_i are $k_i\times k_i$ and $k_i\times k_{i-1}$ matrices, respectively, and Y_i is a $k_i\times 1$ vector. The validity of Eq. (1.15) or (1.16) is verified if we directly differentiate the original relations (1.1) – (1.3) with respect to μ and next carry out operations analogous to those presented in [6].

2. Linear piecewise-continuous systems. Suppose that the initial conditions determining the solutions of the sequence of Eqs. (1.8) depend also on a certain parameter. Then, since the right-hand side of (1.8) does not depend upon this parameter, after an appropriate differentiation we arrive, instead of at (1.16), at a homogeneous sequence of equations

$$y_i^{\cdot} = A_i y_i \sigma(t - t_i) + B_i y_{i-1} \delta(t - t_i), \quad i = \ldots, -1, 0, 1, \ldots \qquad (2.1)$$

The piecewise-continuous system corresponding to (2.1) is obtained, obviously, from the original one after a variation of the initial conditions and, therefore, can be named a system of variational equations. The solutions of this system satisfy the superposition principle, therefore, it is linear and piecewise-continuous (in contrast to piecewise-linear system in the usual sense).

Generally speaking, an arbitrary solution of (2.1) also is continuable only to the side of increase of the argument t. At the same time its solution, which exists for any real t and, consequently (see (1.16)), is determined by the initial condition

$$y_0 \big|_{t=t_*} = \alpha(\mu) \qquad (2.2)$$

can be written in the form

$$y_i = U_i(t, t_*) \alpha \qquad (2.3)$$

Here U_i is the $k_i \times k_0$ matrix solution of (2.1), satisfying the initial condition

$$U_0(t_*, t_*) = E_{k_0} \qquad (2.4)$$

where E_{k_0} is the unit $k_0 \times k_0$ matrix.

If the original solution $x_i(t, \mu)$ is T-periodic in the sense of (1.7), then the matrix coefficients A_i and B_i also are T-periodic in the same sense. Hence it follows that the $k_i \times k_0$ matrix $U_{i+n}(t + T, t_*)$ satisfies system (2.1) and, furthermore, belongs to family (2.3). Therefore, we can write down

$$U_{i+n}(t + T, t_*) = U_i(t, t_*) U_n(t_* + T, t_*) \qquad (2.5)$$

In exactly the same way as in the theory of continuous linear equations with periodic coefficients, from relations (2.5) we can arrive at the characteristic equation

$$| U_n(t_* + T, t_*) - e^{\lambda T} E_{k_0} | = 0 \qquad (2.6)$$

where λ is the characteristic index. To an arbitrary root λ of determinant (2.6) there corresponds a particular solution of (2.1) belonging to family (2.3) which has the form

$$y_i = e^{\lambda T} v_i(t, \mu) \qquad (2.7)$$

where v_i is T-periodic in t in the sense of (1.7). If all k_0 characteristic indices of determinant (2.6) are distinct or have prime elementary divisors, there are k_0 independent particular solutions of type (2.7), whose superposition yields the "general" solution of (2.1).

We introduce into consideration the following ordered sequence of linear systems:

$$z_i^{\cdot} = - z_i A_i [1 - \sigma(t - t_{i+1})] - z_{i+1} B_{i+1} \delta(t - t_{i+1}) \qquad (2.8)$$

Each equation (2.8) serves to define a $1 \times k_i$ vector z_i and, here in contrast to (1.8), (1.16) and (2.1), during the integration we need to assume that $z_i = 0$ for $t > t_{i+1}$. Generally speaking, an arbitrary solution of the linear piecewise-continuous system corresponding to (2.8) is continuable to the side of decreasing t. As to solutions of (2.8) continuable to both sides with respect to t, between them and the similar solutions of (2.1) there is a definite relation. Indeed,

$$\frac{d}{dt}\sum_{i=-\infty}^{\infty} z_i y_i = \sum_{i=-\infty}^{\infty} \{-z_i A_i y_i [1-\sigma(t-t_{i+1})] - z_{i+1} B_{i+1} y_i \delta(t-t_{i+1}) + \\ + z_i A_i y_i \sigma(t-t_i) + z_i B_i y_{i-1} \delta(t-t_i)\} = 0 \qquad (2.9)$$

We integrate this relation with respect to t in the limits from t_* to t. Then, since the writing of (2.1) assumes that $y_i = 0$ for $t < t_i$, while the writing of (2.8) assumes that $z_i = 0$ for $t > t_{i+1}$, we obtain

$$z_i y_i = z_0 y_0 \,|_{t=t_*} \qquad (2.10)$$

The presence of relation (2.10) permits us to consider the system (2.8) as being adjoint relative to (2.1).

Substituting into (2.10) the independent particular solutions (2.1) of form (2.7), we obtain k_0 mutually-independent linear first integrals of (2.8). The inversion of these integrals leads to the forming of the general solution of (2.8), which can be represented in the form of the superposition of particular solutions

$$z_i = e^{-\lambda T} w_i(t, \mu) \qquad (2.11)$$

where w_i is a T-periodic function of t. Hence it follows that to each index λ of system (2.1) there corresponds an index $-\lambda$ of system (2.8). In particular, the number of zero characteristic indices coincide and, consequently, so do the number of periodic solutions of these systems.

When $\mu = 0$ system (1.8) admits of an s-parameter family of T-periodic solutions (1.11), therefore, the system of variational equations (2.1) with $\mu = 0$ admits of s mutually-independent T-periodic solutions $\partial\varphi_i / \partial h_r$ $(r = 1, \ldots, s)$. Correspondingly, the adjoint system (2.8) with $\mu = 0$ also should admit of s T-periodic solutions which we subsequently denote by $z_i^{(r)}$ $(r = 1, \ldots, s)$.

Let us consider the question of the existence of a T-periodic solution of system (1.16) with $\mu = 0$, assuming, furthermore, that into the coefficients of this system and into the inhomogeneous terms we have substituted $x_i^{(0)} = \varphi_i(t, h_1, \ldots, h_s)$. With this aim let us differentiate the quantity

$$\sum_{i=-\infty}^{\infty} z_i^{(r)} u_i^{(0)}, \qquad u_i^{(0)} = u_i /_{\mu=0}$$

with respect to t. Then, by virtue of (1.16) and (2.8), we obtain

$$\frac{d}{dt}\sum_{i=-\infty}^{\infty} z_i^{(r)} u_i^{(0)} = \sum_{i=-\infty}^{\infty}\left[z_i^{(r)}\left(\frac{\partial X_i}{\partial\mu}\right)\sigma(t-t_i) + z_i^{(r)}(Y_i)\,\delta(t-t_i)\right] \qquad (2.12)$$

Here and later the parantheses denote that $\mu = 0$, $t_i = t_i|_{\mu=0}$, $x_i = \varphi_i(t, h_1, \ldots, h_s)$ should be substituted into the corresponding quantity. Integrating this relation with respect to t in the limits from t_0 to t_n and taking into consideration that the writing of (1.16) assumes the validity of the equality $u_i = 0$ for $t < t_i$, and the writing of (2.8) assumes $z_i = 0$ for $t > t_{i+1}$, we obtain the condition for the T-periodicity of $u_i^{(0)}$ in the following form:

$$P_r(h_1, \ldots, h_s) \equiv \sum_{i=1}^{n}\left[\int_{t_{i-1}}^{t_i} z_i^{(r)}\left(\frac{\partial X_i}{\partial\mu}\right)dt + z_i^{(r)}(t_{i-1})(Y_i)_{t=t_{i-1}}\right] = 0 \qquad (2.13)$$

$$r = 1, \ldots, s$$

Thus a function $u_i^{(0)}$, T-periodic in the sense of (1.7), corresponds only to those solutions in family (1.11), whose parameters $h_1,\ldots, h_s$ satisfy the s equations (2.13).

3. Existence of a T-periodic solution. For what follows it is essential that by continuing further to differentiate Eq. (1.16) with respect to μ in full correspondence with the scheme described in Sect. 1, we obtain the equations for determining the successive derivatives $\partial^2 x_i / \partial\mu^2$, $\partial^3 x_i / \partial\mu^3,\ldots$. All these equations are of the same type as (1.16), and their homogeneous parts coincide. When $\mu = 0$ their coefficients have the sense in (1.16) if and only if

$$g_i^{\cdot} /_{\mu=0,[t=t_i} \neq 0 \tag{3.1}$$

Let us assume that this inequality is fulfilled. Then, after the determination of the functions $x_i(t, 0)$ as a result of solving the one-type systems, linear in the intervals of continuity, we can successively determine the functions $u_i^{(0)} = \partial x_i/\partial\mu$, $\partial^2 x_i / \partial\mu^2,\ldots$ and next compose the formal equality

$$x_i(t, \mu) = x_i(t, 0) + \left(\frac{\partial x_i}{\partial\mu}\right)\mu + \frac{1}{2}\left(\frac{\partial^2 x_i}{\partial\mu^2}\right)\mu^2 + \mu^3\ldots \tag{3.2}$$

By virtue of the assumptions made concerning the properties of the original piecewise-continuous system, the series occurring in the right-hand side converge for sufficiently small μ. The proof of this fact is carried out in exactly the same way as in the continuous case.

The switching instants $t_i(\mu)$, necessary for the determination of the piecewise-continuous solution (1.5), are defined by the series

$$t_i(\mu) = t_i(0) + \left(\frac{dt_i}{d\mu}\right)\mu + \frac{1}{2}\left(\frac{d^2 t_i}{d\mu^2}\right)\mu^2 + \mu^3\ldots \tag{3.3}$$

To determine the coefficients of this series it is necessary to differentiate relation (1.2) an appropriate number of times with respect to μ for $t = t_i(\mu)$. Here, obviously,

$$\frac{dt_i}{d\mu} = -\frac{\partial' g_{i+1}}{\partial\mu}(g_{i+1}^{\cdot})^{-1}\Big|_{t=t_i} \tag{3.4}$$

Hence it is clear that a correction of order μ^j ($j = 1, 2,\ldots$) to the instant $t_i(\mu)$ is determined only after the functions $(\partial^j x_i/\partial\mu^j)$ have been found. Let us now return to the solving of the fundamental problem stated in Sect. 1. We seek the existence conditions for a T-periodic solution of the original piecewise-continuous system, which turns into a generating solution belonging to family (1.11) when $\mu = 0$ Here, without loss of generality, we assume that the unknown solution $x_i(t, a, \mu)$ satisfies initial condition (1.6) where $t_* = t_0(0)$ and, consequently,

$$x_0(t_0(0), a, \mu) \equiv a \tag{3.5}$$

Initial conditions (3.5) differ from the initial conditions corresponding to the generating solution by a correction which vanishes together with μ. Therefore,

$$a(\mu) = a(0) + \gamma(\mu) \tag{3.6}$$

where the ($k_0 \times 1$)-dimensional correction $\gamma(\mu)$ tends to zero as $\mu \to 0$. By decreasing μ we can always achieve the required smallness of correction γ, therefore, the T-periodic function $x_i(t, a, \mu)$ can be expanded into a series in powers of μ and γ, which converges for a sufficiently small μ. If, furthermore, the components of correction γ themselves are analytic in μ, then the unknown solution is represented in the form of the following series in powers of μ:

$$x_i(t, \mu) = \varphi_i(t, h_1, \ldots, h_s) + u_i^{(0)}\mu + \mu^2 \ldots \tag{3.7}$$

Here the series coefficients themselves are T-periodic in t and, in particular, $u_i^{(0)}$ is a T-periodic solution of (1.16).

The nature of the dependency of the components of γ on μ, as equally also the very possibility of determining them, emerge during the investigation of the periodicity conditions which are written in the form of the following $k_0 \times 1$ vector equation:

$$\Psi(\gamma, \mu) \equiv x_n(t_0(0) + T, \ a(0) + \gamma, \ \mu) - a(0) - \gamma = 0 \tag{3.8}$$

The structure of Eqs. (3.8) is completely analogous to the structure of the corresponding equations in the theory of periodic solutions of analytic nonautonomous systems with a small parameter in the case of an unisolated generating solution [2]. Here, therefore, by completely analogous reasonings we can show that for a solution $\gamma(\mu)$, analytic in μ, of system (3.8) to exist, the presence of simple solutions in the first-approximation equations

$$\left(\frac{\partial\Psi}{\partial\gamma}\right)\gamma + \left(\frac{\partial\Psi}{\partial\mu}\right)\mu = 0 \tag{3.9}$$

is sufficient. Let us consider these equations in detail. By virtue of (3.8) we can assert that

$$\frac{\partial\Psi}{\partial\gamma} = U_n(t_0 + T, \ t_0) - E_k, \tag{3.10}$$

where $U_i(t, t_0)$ is a $k_i \times k_0$ matrix solution of system (2.1) with a unit initial matrix (2.4) under the condition that $t_* = t_0$. On the other hand,

$$\left(\frac{\partial\Psi}{\partial\mu}\right) = u_n'\,|_{t=t_0(0)+T} \tag{3.11}$$

Here u_i' is the solution of Eq. (1.16) with $\mu = 0$, which, in contrast to $u_i^{(0)}$ is not T-periodic but is determined by virtue of (3.5) by the zero initial conditions

$$u_0'\,|_{t=t_0(0)} = 0 \tag{3.12}$$

We multiply Eq. (3.9) from the left by the $1 \times k_0$ row-vector $z_n^{(r)}(t_0(0) + T)$. As a consequence of (2.4) and (2.10) we have

$$z_n^{(r)}(Y_n)\,|_{t=t_0(0)+T} = z_0^{(r)}\,|_{t=t_0(0)} \tag{3.13}$$

Hence, because of the T-periodicity of the function $z_i^{(r)}$, instead of (3.9) we obtain

$$z_n^{(r)} u_n'\,|_{t=t_0(0)+T} = 0 \tag{3.14}$$

On the other hand, the function u_i' exists for any real t and, consequently, relation (2.12) is valid for this function. By integrating (2.12) in the limits from $t_0(0)$ to $t_0(0) + T$ and taking (3.12) into account, we obtain

$$z_n^{(r)} u_n'\,|_{t=t_0(0)+T} = P_r(h_1, \ldots, h_s) \tag{3.15}$$

Thus, the result of eliminating the components of γ from system (3.9) can be written in the form of s equations in s unknowns

$$P_r(h_1, \ldots, h_s) = 0 \tag{3.16}$$

which coincide completely with Eqs. (2.13). Thus, for a T-periodic solution to exist it is sufficient that system (3.16) admit of a simple solution, i.e. one for which

$$\frac{\partial (P_1, \ldots, P_s)}{\partial (h_1, \ldots, h_s)} \neq 0 \qquad (3.17)$$

4. Stability of a T-periodic solution. The variational equations (2.1) are obtained as a result of a formal differentiation of the original Eqs. (1.8) with respect to a parameter connected with the initial conditions. Therefore, together with (3.1), the following symbolic notation

$$y_i^{\cdot} = \frac{\partial F_i}{\partial x_i} y_i + \frac{\partial F_i}{\partial x_{i-1}} y_{i-1} \qquad (4.1)$$

is also valid, which essentially coincides with the homogeneous part of Eq. (1.13). Here we need to keep in mind that the differentiation with respect to the vector x_i is carried out in exactly the same way as with resepect to a parameter not depending on time. The subsequent investigation of the stability in-the-small of the T-periodic solution being considered is carried out on the basis of variational equations in the form (4.1). This simplifies the calculations significantly and permits us to illustrate visually the analogy between piecewise-continuous systems and systems which are continuous and analytic.

Thus, we turn to the determination of the "critical" particular solutions of form (2.7) of the system of variational equations (4.1), constructed close to the T-periodic solution being considered. Following the method presented in [2], the successive approximations to the critical characteristic indices will be found during the determination of the T-periodic solutions of the system

$$v_i^{\cdot} = \frac{\partial F_i}{\partial x_i} v_i + \frac{\partial F_i}{\partial x_{i-1}} v_{i-1} - \lambda v_i \quad (\lambda\,|_{\mu=0} = 0) \qquad (4.2)$$

In the nonautonomous case being considered, system (4.2) with $\mu = 0$ admits of s independent T-periodic solutions $\partial \varphi_i / \partial h_r$ $(r = 1,\ldots, s)$, while the determinant (2.6) has an s-fold zero root, corresponding to these solutions, with prime elementary divisors. Therefore [2], a T-periodic solution of system (4.2) is analytic in μ, so that

$$v_i = v_i^{(0)} + \mu v_i^{(1)} + \mu^2 \ldots, \qquad \lambda = \lambda_1 \mu + \mu^2 \ldots \qquad (4.3)$$

and in the generating approximation $(\mu = 0)$ we have

$$v_i^{(0)} = \sum_{r=1}^{s} a_r \frac{\partial \varphi_i}{\partial h_r} \qquad (4.4)$$

Here $a_1,\ldots, a_s$ are certain mutually independent constants.

The equation for the T-periodic correction $v_i^{(1)}$ is obtained if we differentiate (4.2) with respect to μ and then set $\mu = 0$, and has the form

$$v_i^{(1)\cdot} = \left(\frac{\partial F_i}{\partial x_i}\right) v_i^{(1)} + \left(\frac{\partial F_i}{\partial x_{i-1}}\right) v_{i-1}^{(1)} + \left(\frac{\partial'}{\partial \mu}\frac{\partial F_i}{\partial x_i}\right) v_i^{(0)} + \left(\frac{\partial'}{\partial \mu}\frac{\partial F_i}{\partial x_{i-1}}\right) v_{i-1}^{(0)} - \lambda_1 v_i^{(0)} \qquad (4.5)$$

We recall that the prime here, as also earlier, signifies total partial differentiation, while the parentheses signify that the corresponding quantity is computed in the generating approximation. We note further that by virtue of the original equations of motion (1.8) we have

$$\frac{\partial'}{\partial \mu}\frac{\partial F_i}{\partial h_r} = \frac{d}{dt}\frac{\partial^2 x_i}{\partial h_r \partial \mu} = \frac{d}{dt}\frac{\partial u_i}{\partial h_r} \qquad (4.6)$$

Here we have in mind the solution $x_i\ (t,\ h_1,\ldots,\ h_s,\ \mu)$, of the original system, which is T-periodic only if the constants $h_1, \ldots, h_s$ satisfy system (3.16).

On the other hand,

$$\frac{\partial'}{\partial\mu}\frac{\partial F_i}{\partial h_r} = \frac{\partial'}{\partial\mu}\left(\frac{\partial F_i}{\partial x_i}\frac{\partial x_i}{\partial h_r} + \frac{\partial F_i}{\partial x_{i-1}}\frac{\partial x_{i-1}}{\partial h_r}\right) = \frac{\partial'}{\partial\mu}\frac{\partial F_i}{\partial x_i}\frac{\partial x_i}{\partial h_r} + \frac{\partial'}{\partial\mu}\frac{\partial F_i}{\partial x_{i-1}}\frac{\partial x_{i-1}}{\partial h_r} + \frac{\partial F_i}{\partial x_i}\frac{\partial u_i}{\partial h_r} + \frac{\partial F_i}{\partial x_{i-1}}\frac{\partial u_{i-1}}{\partial h_r} \quad (4.7)$$

Therefore, keeping (4.4) in mind, Eq. (4.5) can be rewritten as

$$v_i^{(1)\cdot} = \left(\frac{\partial F_i}{\partial x_i}\right)v_i^{(1)} + \left(\frac{\partial F_i}{\partial x_{i-1}}\right)v_{i-1}^{(1)} + \sum_{r=1}^{s}\left[\frac{d}{dt}\left(\frac{\partial u_i}{\partial h_r}\right) - \left(\frac{\partial F_i}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial h_r}\right) - \left(\frac{\partial F_i}{\partial x_{i-1}}\right)\left(\frac{\partial u_{i-1}}{\partial h_r}\right) - \lambda\frac{\partial \varphi_i}{\partial h_r}\right] a_r \quad (4.8)$$

By virtue of the last equation, since in its right-hand side the quantity $\partial F_i / \partial x_{i-1}$ can be replaced by $B_i\delta(t - t_i)$, we obtain

$$\frac{d}{dt}\sum_{i=-\infty}^{\infty} z_i^{(r)} v_i^{(1)} = \sum_{q=1}^{s} a_q \frac{d}{dt}\sum_{i=-\infty}^{\infty} z_i^{(r)}\left(\frac{\partial u_i}{\partial h_q}\right) - \sum_{q=1}^{s} a_q \sum_{i=-\infty}^{\infty}\left[z_i^{(r)\cdot} + z_i^{(r)}\left(\frac{\partial F_i}{\partial x_i}\right) + z_{i+1}^{(r)}\left(\frac{\partial F_{i+1}}{\partial x_i}\right)\right]\left(\frac{\partial u_i}{\partial h_q}\right) - \lambda_1 a_r. \quad (4.9)$$

In the derivation of relation (4.9) the summation index r in Eq. (4.8) is changed to q and the order of summation over i in the third term inside the brackets is shifted by unity ($i - 1$ is replaced by i). Furthermore, fulfillment was assumed of the orthogonality and norming conditions

$$\sum_{i=-\infty}^{\infty} z_i^{(r)}\frac{\partial \varphi_i}{\partial h_q} = \delta_{qr} \quad (4.10)$$

where δ_{qr} is the Kronecker symbol. In relation (4.9) the brackets include the expression coinciding with (2.8) for $t < t_{i+1}$. The writing of (2.8) assumes that $z_i^{(r)} = 0$ for $t > t_{i+1}$. Therefore, the corresponding term from (4.9) vanishes. Consequently, by integrating (4.9) in the limits from $t_0(0)$ to $t_n(0) = t_0(0) + T$, we obtain

$$\sum_{i=-\infty}^{\infty} z_i^{(r)} v_i^{(1)}\Big|_{t_0}^{t_0+T} = \sum_{q=1}^{s} a_q \sum_{i=-\infty}^{\infty} z_i^{(r)}\left(\frac{\partial u_i}{\partial h_q}\right)\Big|_{t_0}^{t_0+T} - \lambda_1 T a_r \quad (4.11)$$

By virtue of the T periodicity of the functions $z_i^{(r)}$ and $u_i^{(0)}$, with due regard to (2.12) we have

$$\sum_{i=-\infty}^{\infty} z_i^{(r)}\left(\frac{\partial u_i}{\partial h_q}\right)\Big|_{t_0}^{t_0+T} = \frac{\partial}{\partial h_q}\sum_{i=-\infty}^{\infty} z_i^{(r)} u_i^{(0)}\Big|_{t_0}^{t_0+T} = \frac{\partial P_r}{\partial h_q} \quad (4.12)$$

Consequently, the condition for the T periodicity of function $v_i^{(1)}$ is written as

$$\sum_{q=1}^{s}\frac{\partial P_r}{\partial h_q} a_q = \lambda_1 T a_r \quad (4.13)$$

Thus, similarly to what holds in the continuous case [2], in order for the T-periodic solution being considered of the nonautonomous piecewise-continuous system to be stable, it turns out to be sufficient that all the roots $\lambda_1^{(1)},\ldots,\lambda_1^{(s)}$ of the s th-degree equation

$$\left|\frac{\partial P_r}{\partial h_q}-\lambda_1 T\delta_{qr}\right|=0 \tag{4.14}$$

have negative real parts. It can be shown that a similar correspondence is preserved also in the autonomous case as well as in the more complex cases when the critical characteristic indices for $\mu = 0$ have nonprime elementary divisors.

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ON STEADY CAPILLARY-GRAVITATIONAL WAVES OF FINITE AMPLITUDE AT THE SURFACE OF FLUID OVER AN UNDULATING BED

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The problem of steady capillary-gravitational plane waves of finite amplitude at the surface of a stream of perfect incompressible fluid over an undulating bed under constant surface pressure is considered. The intersection of the undulating bed surface with the vertical plane is assumed to be a periodic curve which is called the bed line and specified by an infinite trigonometric series. An exact solution of this problem, which reduces it to a system of nonlinear integral and transcedental equations, is presented. The theorem of existence and uniqueness of solution of that system is obtained on the assumption of smallness of the bed line amplitude. The method of proving this theorem is indicated and the method of deriving solutions with any degree of approxima-